Math 32A, Lecture 1
Multivariable Calculus

Sample Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: Solutions

UID: ____________________

Section: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Consider the plane curve $\mathbf{r}(t) = \langle t^2 - 2t, e^t \rangle$.

(a) [4pts.] Calculate the curvature of $\mathbf{r}(t)$ at $t = 1$.

(b) [5pts.] Find the osculating circle to $\mathbf{r}(t)$ at $t = 1$. [Hint: This can be done without any additional differentiation.]

(c) [1pts.] Explain, in words, what the osculating circle represents geometrically.

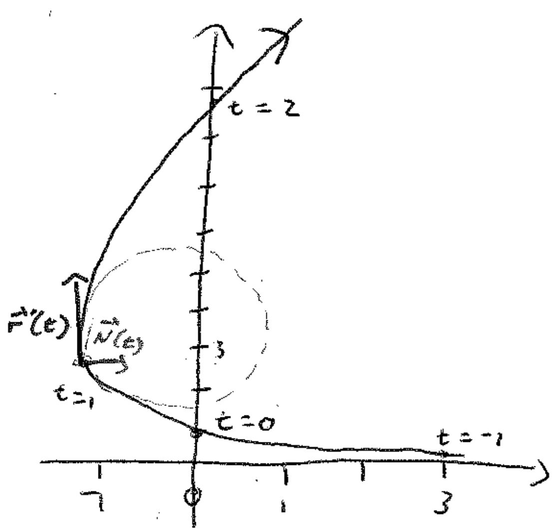
(a) $\vec{r}(t) = \langle t^2-2t, e^t, 0\rangle$ $\qquad$ $\vec{r}(1) = \langle -1, e, 0\rangle$

$\vec{r}\,'(t) = \langle 2t-2, e^t, 0\rangle$ $\qquad$ $\vec{r}\,'(1) = \langle 0, e, 0\rangle$

$\vec{r}\,''(t) = \langle 2, e^t, 0\rangle$ $\qquad$ $\vec{r}\,''(1) = \langle 2, e, 0\rangle$

$$\kappa(1) = \frac{\|\vec{r}\,'(1) \times \vec{r}\,''(1)\|}{\|\vec{r}\,'(1)\|^3} = \frac{\|\langle 0, 0, -2e\rangle\|}{e^3} = \frac{2e}{e^3} = \frac{2}{e^2}.$$

(b) Since $\vec{r}(t)$ lies in the xy-plane, the unit normal $\vec{N}(1)$, which is perpendicular to $\vec{r}\,'(1) = \langle 0, e, 0\rangle$, is $\langle \pm 1, 0, 0\rangle$. Inspection of the graph indicates it is $\langle 1, 0, 0\rangle$. So the osculating circle has radius $\frac{1}{\kappa} = \frac{e^2}{2}$ and is centered at



$$\vec{r}(1) + \frac{1}{\kappa}\vec{N} = \langle -1, e, 0\rangle + \langle \tfrac{e^2}{2}, 0, 0\rangle = \langle \tfrac{e^2-2}{2}, e, 0\rangle$$

$\rightsquigarrow$ it is

$$\left(x - \left(\frac{e^2-2}{2}\right)\right)^2 + (y-e)^2 = \frac{e^4}{4}$$

(c) The osculating circle is the circle that most closely approximates the curve at $\vec{r}(1)$

Problem 2.

Consider the space curve $\mathbf{r}(t) = \langle 2t, 4t^{\frac{3}{2}}, 2t^{\frac{3}{2}} \rangle$.

(a) [5pts.] Find the length of this curve over the interval $0 \le t \le 1$.

(b) [5pts.] Find a reparametrization of this curve such that the speed of the curve at ~~time $t = 1$~~ is 21.

$(2, 4, 2)$

(a) $\vec{r}\,'(t) = \langle 2, 6t^{1/2}, 3t^{1/2} \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{4 + 36t + 9t}$

$= \sqrt{4 + 45t}$

$s = \int_0^1 \sqrt{4 + 45t}\, dt$

$= \frac{1}{45}\left(\frac{2}{3}\right)(4 + 45t)^{3/2}\Big|_0^1$

$= \frac{2}{135}\left[(49)^{3/2} - (4)^{3/2}\right]$

$= \frac{2}{135}[343 - 8]$

$= \frac{2}{135}(335)$

$= \boxed{\frac{134}{27}}$

(b) Presently, the speed at time $t = 1$ is $\|\vec{r}\,'(1)\| = \sqrt{4 + 45(1)} = \sqrt{49} = 7$. Let $t = 3u$ be our reparametrization, so that $\vec{r}(u) = \langle 6u, 4(3u)^{3/2}, 2(3u)^{3/2} \rangle$.

Then $\vec{r}\,'(u) = \langle 6, 18(3u)^{1/2}, 6(3u)^{1/2} \rangle = \underbrace{3}_{\frac{dt}{du}} \langle 2, 6(3u)^{1/2}, (3u)^{1/2} \rangle$.

We see that at $u = \frac{1}{3}$ (corresponding to $t = 1$), $\|\vec{r}\,'(u)\| = 21$.

Problem 3.

Decide whether each of the following limits exist, and if they do, compute them. Justify your answers.

(a) [3pts.] $\lim_{(x,y)\to(4,2)} \frac{y-2}{\sqrt{x^2-4}}$.

(b) [3pts.] $\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+2y^2}$.

(c) [4pts.] $\lim_{(x,y)\to(0,0)} \tan x \sin\left(\frac{1}{|x|+|y|}\right)$.

(a) This function is cts at $(4,2)$, since $y-2$ and $\sqrt{x^2-4}$ both are, and $\sqrt{x^2-4} \neq 0$ at $x=4$.

So $$\lim_{(x,y)\to(4,2)} \frac{y-2}{\sqrt{x^2-4}} = \frac{2-2}{\sqrt{16-4}} = \frac{0}{\sqrt{12}} = 0.$$

(b) We choose two paths to $(0,0)$:

Along $x=y$: $$\lim_{x\to 0} \frac{x^2}{3x^2+2x^2} = \lim_{x\to 0} \frac{1}{5} = \frac{1}{5}$$

Along $x=0$: $$\lim_{y\to 0} \frac{0}{2y^2} = \lim_{y\to 0} 0 = 0$$

Limit cannot exist.

(c). Observe that $-|\tan x| \leq |\tan x \sin\left(\frac{1}{|x|+|y|}\right)| \leq |\tan x|$ and $\lim_{(x,y)\to(0,0)} |\tan x| = 0$. So by the squeeze thm,

$$\lim_{(x,y)\to(0,0)} \tan x \sin\left(\frac{1}{|x|+|y|}\right) = 0.$$

Problem 4.

Consider the quadric surface $16y^2 - 9x^2 - z^2 = -144$.

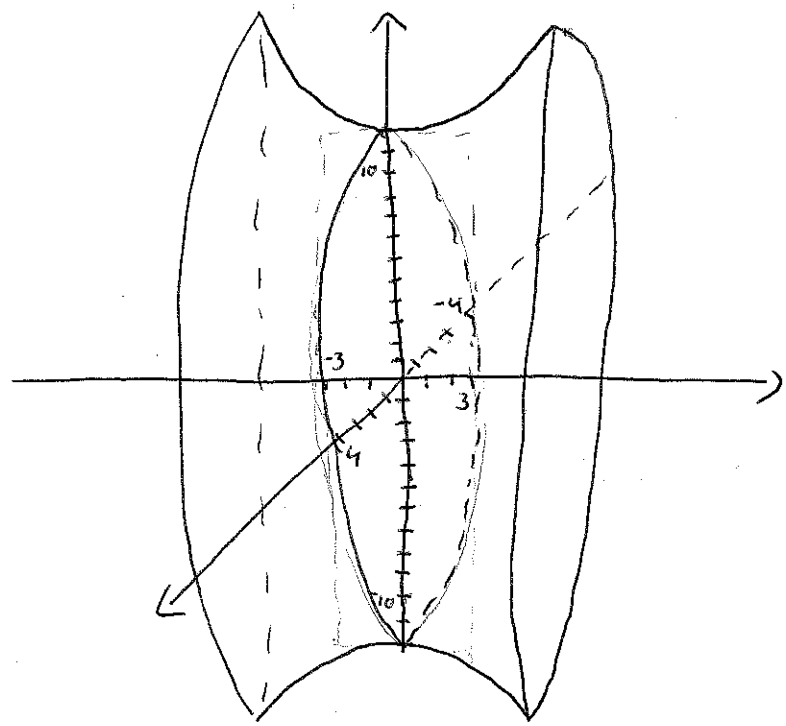
(a) [5pts.] Draw this surface. Be sure to clearly label your axes.

(b) [5pts.] Give a parametrization of the intersection of this surface and the hyperbolic cylinder $25y^2 = z^2 + 25$.

ⓐ $16y^2 - 9x^2 - z^2 = -144$

$$\underbrace{-\left(\frac{y}{3}\right)^2 + \left(\frac{x}{4}\right)^2 + \left(\frac{z}{12}\right)^2 = 1}$$

Hyperboloid of one sheet



ⓑ $z^2 = 25y^2 - 25$

$$16y^2 - 9x^2 - (25y^2 - 25) = -144$$

$$-9y^2 - 9x^2 = -169$$

$$9x^2 + 9y^2 = 169$$

$$x^2 + y^2 = \left(\frac{13}{3}\right)^2$$

$$\begin{cases} x = \frac{13}{3}\cos\theta \\ y = \frac{13}{3}\sin\theta \end{cases}$$

$$z^2 = 25\left(\frac{13}{3}\sin\theta\right)^2 - 25$$

$$z = \pm\sqrt{25\left(\frac{169}{9}\right)\sin^2\theta - 25}$$

$$\begin{cases} x = \frac{13}{3}\cos\theta \\ y = \frac{13}{3}\sin\theta \\ z = \pm\sqrt{25\left(\frac{169}{9}\right)\sin^2\theta - 25} \end{cases}$$

Note that not every value of θ has a solution.

Problem 5.

Consider the multivariable function $f(x, y) = \sqrt{4x - 3y^2}$.

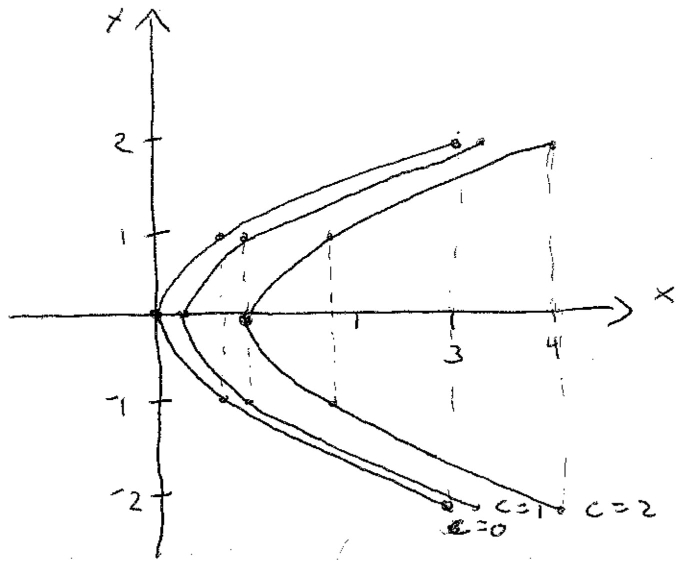
(a) [5pts.] Draw a contour map of f, using the constants $c = 0, 1$, and 2.

(b) [5pts.] Find the partial derivatives of f at $(1, 1)$. Indicate what the signs of these numbers tell you about the contour map you drew.

(a) $c = \sqrt{4x - 3y^2}$

$c^2 = 4x - 3y^2$

$x = \dfrac{3y^2 + c^2}{4}$



(b) $F_x(x,y) = \dfrac{4}{2\sqrt{4x - 3y^2}} = \dfrac{2}{\sqrt{4x-3y^2}}$ $\quad f_x(1,1) = 2$

$f_y(x,y) = \dfrac{-3y}{\sqrt{4x-3y^2}}$ $\quad f_y(1,1) = -3$

Notice that at $(1,1)$, for z to increase (i.e. to move from $c=1$ to $c=2$, and so on) we have to increase x (move to the right) or decrease y (move down).